Lemma 9.2.7

Let $X \in \mathcal{X}$, $l \geq 0$, $K = (i_1, \dots, i_l) \in I^l$

(1) Assume that $\Delta^{Y\mathrm{re}} \subseteq N_0^I \cup -N_0^I$ for any $Y = r_{i_k} \cdots r_{i_1}(X)$ $\boxed{0 \leq k < l}$

$k=0.\ k<l$

Then TFAE:

(a) K is X-reduced

(b) $\beta_1^{X,K}, \dots, \beta_l^{X,K}$ are pairwise distinct elements in N_0^I

(c) $\Lambda^X(K) \subseteq N_0^I$

Proof: (a) ⇒ (b) by induction:

if $l \leq 1$ (b) is trivial.

Assume that $l > 1$. let $Z = r_{i_1}(X)$

then $K' = (i_2, \dots, i_l)$ is Z-reduced.

By induction hypothesis:

$\beta_1^{Z,K'}, \dots, \beta_{l-1}^{Z,K'}$ are pairwise distinct in N_0^I.

Since K is X-reduced,

they are contained in $\Delta_+^{Z\mathrm{re}} \setminus \{\alpha_{i_1}\}$

Hence by lemma 9.1.19 (1)

$$\beta_1^{X,K} = \alpha_{i_1},\ \beta_2^{X,K} = s_{i_1}^Z(\beta_1^{Z,K'})$$
$$\vdots$$
$$\beta_l^{X,K} = s_{i_1}^Z(\beta_{l-1}^{Z,K'})$$

are pairwise distinct in N_0^I.

(b) ⇒ (c) obviously.

(c) ⇒ (a)

~~Proof of K~~

(2) Let $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$

Assume that $\delta^{Y\mathrm{re}} \subseteq N_0^I \cup -N_0^I$ for any $0 \leq k < l$ for any $Y = r_{i_k} \cdots r_{i_1}(X)$

if K is X-reduced then

$$\delta^{X\mathrm{re}}(w) = \Lambda^X(K)$$

Proof of (2):

By induction:

If $l = 0$, then $w = \mathrm{id}_X$ $\delta^{X\mathrm{re}}(w) = \phi = \Lambda^X(K)$

Assume $l > 0$ and K is X-reduced.

Then $(i_1, \dots, i_{l-1})$ is X-reduced.

Let $w' = \mathrm{id}_X s_{i_1} \cdots s_{i_{l-1}}$,

then $w = w' s_l$

$$\delta^{X\mathrm{re}}(w') \underset{\text{归纳假设}}{=} \{\beta_1^{X,K}, \dots, \beta_{l-1}^{X,K}\}$$

and $w'(\alpha_{i_l}) = \beta_l^{X,K} \in N_0^{I}$ — by (1) (c)

Hence $\Delta^{X\,re}(w) = \Delta^{X\,re}(w') \cup \{\beta_l^{X,K}\}$

$= \Lambda^{X}(K)$

lem 9.1.21 (2)

Lemma 9.2.8

Let $X \in \mathcal{X}$, $l \geq 0$, $K = (i_1, \dots, i_l)$

Assume that K is X-reduced, $|\Delta_+^{X\,re}| > l$,

$\Delta^{r_{i_k} \cdots r_{i_1}(X)\,re} \subseteq N_1^{I} \cup N_0^{I}$, $\forall\, 0 \leq k \leq l$.

then $\exists\, i$, s.t $(i, i_1, \dots, i_l)$ is $r_i(X)$-reduced

Proof: Let $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$

$(i_1, \dots, i_l, i_{l+1})$

$\Lambda + 1$

Then $\Delta^{X\,re}(w) \neq \Delta_+^{X\,re}$

then $\exists\, i$, s.t $\alpha_i \notin \Delta^{X\,re}(w) = \Lambda^{X,K}$

Hence the claim follows from lemma 9.2.2 (2). □

Remark 9.2.9 $\overline{m}_{ij}^{X}$

Let $i, j \in I$, $i \neq j$, $K = (i_k)_{k \geq 1}$ $= (i, j, i, \dots)$

Let $X \in \mathcal{X}$.

then any X-reduced sequence with entries $\{i, j\}$ and starting with i is a beginning of K.

Let K_{ij}^{X} be the longest X-reduced beginning of K, if it exists, and K otherwise

We write $\overline{m}_{ij}^{X}$ for the length of K_{ij}^{X}.

Clearly, $\overline{m}_{ij}^{X} \geq 2$ ~~(lem 9.2.5 (2))~~

(lem 9.2.5 (2))

(i, j)

For any $X \in \mathcal{X}$, any $i, j \in I$, $i \neq j$

let $\tau(X, i, j) = (r_i(X), j, i)$ $\quad$ $X \times I \times I$ $\quad$ $i \neq j$

$\sigma(X, i, j) = (X, j, i)$

Clearly, $\sigma^2(X, i, j) = (X, i, j)$

$\tau^{-1}(X, i, j) = (r_j(X), j, i)$

$= \boxed{\sigma \tau \sigma}(X, i, j)$

— —, generated by τ and σ as a monoid

In prop 9.2.14

we will prove $\bar{m}_{ij}^{X}$ is constant

on the orbits of this group in — — case.

(沿用上面的记号)

(CG3') For any $X \in \mathcal{X}$, any X-reduced K,

$\Lambda^X(K) \subseteq \mathbb{N}_0^I$.

(CG4') For any $X \in \mathcal{X}$, any $i, j \in I$, $i \neq j$, $\bar{m}_{ij}^{X} < \infty$

we have $(r_j r_i)^m(X) = X$, $(m = \bar{m}_{ij}^{X})$

$\mathrm{id}_X (s_i s_j)^m(\alpha_k) = \alpha_k$

for all $k \in I \setminus \{i, j\}$

Lemma 9.2.10

Let $X \in \mathcal{X}$, $l \geq 2$, $K = (i_1, \ldots, i_l) \in I^l$.

$1 \leq m < n \leq l$

Assume that $i_m = i_n$ and $i_k \neq i_m$ $\forall m < k < n$

Then $\exists a_{m+1}, \ldots, a_{n-1} \in \mathbb{N}_0$ s.t

$$\beta_m^{X,K} + \beta_n^{X,K} = \sum_{k=m+1}^{n-1} a_k \beta_k^{X,K}$$

Proof: Induction by $n-m$

If $n = m+1$, ✓

$\beta_m^{X,K} = -\beta_n^{X,K}$ $\quad$ $\mathbb{Z}^I$

If $n > m+1$, then

$$\beta_m^{X,K} + \beta_n^{X,K} = \beta_m^{X,K} + id_X s_{i_1} \cdots \boxed{s_{i_{n-1}}(\alpha_{i_n})}$$

$$= \underline{\beta_m^{X,K}} + \underline{id_X s_{i_1} \cdots s_{i_{n-2}}(\alpha_{i_n}} + a_{n-1}\alpha_{i_{n-1}})$$

$a_{n-1} \in \mathbb{N}_0$

$$= \underline{\beta_m^{X,K'}} + \underline{\beta_{n-1}^{X,K'}} + \underline{a_{n-1}}\, \underline{\beta_{n-1}^{X,K}}$$

$K' = (i_1, \ldots, i_{n-2}, i_n)$

- - - -

Since $\beta_k^{X,K'} = \beta_k^{X,K}$ for any $\underline{m \le k < n-1}$

□

Lemma 9.2.11

Let $X \in \mathcal{X}$, $l \geq 3$, $K = (i_1, \ldots, i_l)$

Assume that $i_{l-2} = i_l$ and $\beta_{l-1}^{X,K} = \alpha_j$ for some $j \in I$.

Then $\underline{\beta_{l-2}^{X,K}} \notin \mathbb{N}_0^I$ or $\boxed{\beta_l^{X,K}} \notin \mathbb{N}_0^I$.

Proof:

If $\underline{i_{l-1} = i_l}$

then $\beta_l^{X,K} = -\beta_{l-1}^{X,K} = -\alpha_j$ and the claim is proved.

Assume that $i_{l-1} \neq i_l$

then $\exists\, a \in \mathbb{N}_0$, s.t. $\beta_{l-2}^{X,K} + \beta_l^{X,K} = a\alpha_j$

Since $\beta_{l-1}^{X,K} = \alpha_j$

Remarks 9.1.16(2) and 9.2.6(2)

implies that $\left(\underbrace{\beta_l^{X,K} \notin \mathbb{N}_0 \alpha_j.}_{\pm 1} \right)$

$\beta_l^{X,K} = \beta_{l+}^{X,K}$

是不可能的.

用反证法

假设 $\beta_{l-2}^{X,K} \in N_j^I$ 且 $\beta_l^{X,K} \in N_j^I$

$$\beta_{l-2}^{X,K} = a\alpha_j - \beta_l^{X,K} \notin N_j^I$$

$\square$

Lemma 9.2.12

Assume that $\exists\, X \in \mathcal{X}$, and X-reduced K

s.t $\Lambda^X(K)$ contains an elmt in $(-N_j^I)$.

Then $\exists\, Y \in \mathcal{X}$, Y-reduced K',

s.t $\underline{\Lambda^Y(K')}$ contains an elmt in

$\mathbb{Z}^I \setminus (N_j^I \cup N_0^I)$

Proof.

Assume that $\Lambda^Z(K'')$ contains no elmts in $(-N_j^I)$ for any point Z and any Z-reduced K'' of length $< l$,

and assume that $\beta_{l}^{X,K} \in \underline{-N_j^I}.$

解释： 所有满足 $\Lambda^X(K)$ 中有一个 $-N_j^I$ 里元素

的pair $\underline{(X, K)}$

选出最早出现 $-N_j^+$ 里元素的

那个 (x, k)

Then $l > 1$

Moreover, $\beta_l^{x,k} \neq -\alpha_{i_1}$ since K is reduced

Thus $\beta_l^{x,k} \notin \mathbb{Z}\alpha_{i_1}$

Since $(i_2, \dots, i_l)$ is $r_{i_1}(x)$-reduced

(因为 $(i_1, \dots, i_l)$ is x-reduced)

we conclude that

$$\Lambda^{r_{i_1}(x)}(i_2, \dots, i_l) \ni s_{i_1}^{x}(\beta_l^{x,k}) = \beta_l^{x,k} + b\alpha_{i_1}, \quad b \in \mathbb{Z}$$

$\in \mathbb{Z}^I \setminus (N_j^+ \cup -N_j^+)$

由假设，它不可能在 $-N_j^+$

而 $\beta_l^{x,k} \in -N_j^+$ 且不是 α_{i_1} 的倍数

— — — $\square$

Lemma 9.2.13

Assume that $|I| \geq 2$

Let $Y \in X$, $i, j \in I$, $i \neq j$.

$K = (j, i, j, i, \dots) = (j_1, \dots, j_m)$

be a Y-reduced beginning of K_{ij}^{Y} with $m \geq 2$.

if $\underline{\overline{m}_{ij}^{r_i(Y)}} = m$, then $\beta_m^{Y,K} = \alpha_i$

Proof: Let $K' = (i, j_1, \dots, j_m)$

Since $\overline{m}_{ij}^{r_i(Y)} \leq m$, K' is not $r_i(Y)$-reduced

Hence $\beta_p^{r_i(Y),K'} = -\beta_q^{r_i(Y),K'}$

for some $1 \leq p < q \leq m+1$.

Since $\underline{\beta_{k+1}^{r_i(Y),K'} = s_i^Y(\beta_k^{Y,K})}$ $\forall\ 1 \leq k \leq m$

and since K is Y-reduced

$\Rightarrow p = 1$

$s_i^Y \beta_{p-1}^{Y,K} = s_i^Y \beta_{q-1}^{Y,K}$

$1 \leq p < q \leq m+1$

Since K is Y-reduced

只能 $\underline{p=1}$.

Then $\underline{q = m+1}$, since $\overline{m}_{ij}^{r_i(Y)} = m$.

$\Rightarrow \alpha_i = -s_i^Y(\beta_m^{Y,K}) \Rightarrow \beta_m^{Y,K} = \alpha_i$

□